Sequential functions

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Programme 1

Abstract: We introduce SK-domains and define S-functions which generalise the notion of sequential functions which has been previously introduced by Milner, by Vuillemin and by Kahn andPlotkin; we then show that SK-domains and S-functions constitute a Λ - category .

Fonctions séquentielles

Resumé :Nous généralisons la notion de fonction séquentielle, introduite par Milner, Vuillemin et par Kahn et Plotkin, à des domaines non concrets. Nous obtenons de cette façon une L-catégorie c-à-d une catégorie cartésienne close ordre- enrichie par l'ordre de Scott.

I) INTRODUCTION

The sequentiality problem is highly related with the full abstraction question for models of λ-calculus based languages. In [5] Milner posed the problem in the following form: " One would like to find a concept of sequential continuous function, and show that the model of sequential functions exists and yields a fully abstract interpretation".

Actually the sequentiality problem was originally raised in its most typical form and shown difficult by Plotkin in [7]; there, the model derived from Scott's of continuous functions was shown to be non fully abstract for the language PCF (a typed λ-calculus together with arithmetic and boolean operators).

Indeed Plotkin exhibited two PCF terms which are operationally equivalent but are denoted by two functions differing on an argument which is undefinable in the language. Such an argument is typically the "parallel or" function, a binary boolean function which yields value true, as soon as one of its arguments is true.

The reason of this failure of full abstraction is essentially an inadequate treatment of sequentiality: it is known that PCF-like languages can be evaluated sequentially; therefore in order to provide fully abstract models for such languages, one should restrict to sequential continuous functions (since it is known that the model of continuous functions is complete). On this way, Milner and Vuillemin proposed two different definitions of sequentiality (see[5] and[8]).

Unfortunately their definitions rely on the product structure of the input space of functions and are adequate only for first order functions (not for functionnals).

In order to solve the problem, Kahn and Plotkin[3] proposed a more general definition which abstracts from the product structure of input domain.

Their proposition relies on the axiomatisation of the notion of place in a particular class of domains called concrete domains; nevertheless it cannot be used to provide the fully abstract model of PCF since the category of concrete domains and sequential functions (as defined by Kahn and Plotkin) is not cartesian closed. Notice however that substituting the notion of sequential algorithm for that of function and relaxing Scott's order allow to provide PCF with a fully abstract but non extensional model (see[1,2]).

Attacking the problem led Berry to define the notion of stability (see[1]). Intuitively stable functions are continuous functions together with a particular minimality property. Formally, a function f is stable if for any element x and for any approximation β of f(x), there exists an approximation m(f,x,β) of x s.t.: for any approximation y of x, β approximates f(y) if and only if m(f,x,β) approximates y. A typical example of a non stable function is the parallel-or function since m(por,(tt,tt),tt) does not exist.

These functions, together with rather standard domains, provide a cartesian closed category which cannot be order-enriched with Scott's order and thus cannot provide the fully abstract model of PCF since Milner has shown that the function-based fully abstract model of PCF is unique (up to isomorphism) and is extensional (ie: ordered with Scott's order). More crucially, Berry left the problem open by exhibiting a function which is stable but not sequential; such a function is the least continuous 3-ary boolean function f such that f(tt,ff,$\perp$) = f(ff,$\perp$,tt) = f($\perp$,tt,ff) = tt. Indeed this function has no sequentiality index at ($\perp$,$\perp$,$\perp$) for value tt.

The challange of this paper is to close the sequentiality question, at least in its first part: Our goal is to define a concept of sequential function which could be used to provide an order-extensional model of typed λ-calculus based languages (such as PCF).

In section II we introduce the SK-domains, which are particular bi-ordered structures. The two order relations are intended to compare the behaviour of individuals extensionally and intensionally as in the bi-structures used by Berry. However, contrary to Berry's approach our intensional relation does not impose any significant structure on domains; it only allows to generalise Khan and Plotkin's notion of place in a wide class of domains (not only the concrete ones). In section III we use the intensional preorder and define the S-functions which are continuous functions satisfying those conditions that are intuitively required to get sequential behaviours. We notice that S-functions are not nessarily stable functions: for example, considering the complete lattice of boolean values, the function "def" which is strict and yieds value true for the defined and over defined arguments is an S-function though non stable. Finally in section IV we show that SK-domains together with S-functions constitute a cartesian closed category order-enriched with Scott's order.

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II) SK-DOMAINS

This section introduces the spaces and structures used throughout this paper. Let us precise our terminology and recall the following well known facts:

Let $< D,\leq >$ be a partial order (usually called po and freely denoted D), let $X \subseteq D$ and let $x,y,z \in D$ then we say:
- x is *dominated* -resp. *strictly dominated*- by y (or y *dominates* -resp.*dominates strictly* x) iff: $x \leq y$ -resp. $x<y$ -
- x and y are *comparable* (notation: $x \Diamond y$) iff: $x \leq y$ or $y \leq x$
- z is the *lub (least upper bound)* of X (notation: $z = \vee X$) iff z dominates any element in X and is dominated by any element in D wich dominates all elements of X.
- z is the *glb (greatest lower bound)* of X (notation: $z = \wedge X$) iff z is dominated by all elements of X and dominates any element in D wich is dominated by all elements of X.
- x is the *maximum élément* (notation: $\top$)- resp.*minimum élément* (notation: $\perp$) - of D iff: $x = \vee D$ -resp. $x = \wedge D$ -

We need also the following well known concepts:

2.1 definition: let D be po , X a subset of D, and x an element of D; then:
- X is said to be *directed* if it is non empty and st: $\forall\, \alpha,\beta \in X, \exists\, \gamma \in X: \alpha \leq \gamma \geq \beta$
- x is a *finite* (or isolated or compact) element of D if $\forall\, X \subseteq D$, whenever X is directed, then $(x \leq \vee X) \Rightarrow (\exists\, \alpha \in X: x \leq \alpha)$. ■

2.2 definition: let D be a po; then D is a *complete lattice* if it has a glb and a lub for any of its subset. A complete lattice is *algebraic* if the subset of all finite elements dominated by an element x, is directed and has lub x. If any element dominates only a denumerable set of finite elements then it is *ω-algebraic* ■

Henceforth, we refer to ω-algebraic lattices as *domains*. We denote $\mathfrak{B}(D)$ the set of finite elements of a domain D, and $\mathfrak{B}(x)$ the set of finite elements dominated by x.
In the following, it will be necessary to compare the intensions of elements, abstracting from their extensions. For that purpose, we introduce an equivalence relation on the set of finite elements, and set two elements to be intensionally related if the sets of equivalence classes of the finite elements they dominate are related (via the subset relation). More precisely we introduce the following:

2.3 definition: *SK-domains* are bi-ordered structures $< D, \leq, \sim >$ st:

i) $< D, \leq >$ is a domain

ii) $\sim$ is an equivalence relation on the set of finite elements of $< D, \leq >$. ∎

From now on, whenever unambiguous, an SK-domain $< D, \leq, \sim >$ will be freely denoted D. The partial order "$\leq$" is called *extensional*. The equivalence relation "$\sim$" is called *intensional.*

Given an SK-domain its intensional equivalence relation can be extended and generalised as a preorder relation over the whole domain:

2.4 lemma: let $< D, \leq, \sim >$ be an SK-domain .The relation $x \lesssim y$ defined by: $\mathbf{B}(x)/\sim \subseteq \mathbf{B}(y)/\sim$, is a preorder relation which is coarser than $\leq$.

proof: "$\lesssim$" is clearly a preorder. It is clear also that it is coarser than relation $\leq$ since in a domain, $x \leq y$ iff $\mathbf{B}(x) \subseteq \mathbf{B}(y)$) ∎

Summarizing, an SK-domain is a ω-algebraic lattice together with an intensional preorder which is coarser than the extensional one. Actually this intensional preorder relation is needed only in order to define morphisms of SK-domains and will always be defined via an equivalence relation provided with the set of the finite elements of the domain.

In our case, we are dealing with the problem of generalizing the notion of "the place of an argument" of a function, so as to leave away any reference to the product structure of the input domain. For illustration purposes we consider first a particular class of SK-domains called *basic*, and which contains those SK-domains whose underlying domain is a product of flat domains. They will serve to relate the notion of place used in Kahn-Plotkin's definition of sequentiality and our definition (at least in those SK-domains that are closest to concrete domains). In a further step, we introduce an intensional preorder for the whole class of SK-domains, in particular those whose underlying domains are functional ones. Before, we need the following notions:

2.5 definition: let $D=D_1 \times D_2 \times \ldots \times D_n$ be a basic SK-domain and let $x=<x_i>_{i\leq n}$, $y=<y_i>_{i\leq n}$ be elements of D. Then x and y are *properly compatible* (notation $x \Uparrow y$) if their lub does not contain any maximum element as component ie: $x \Uparrow y \Leftrightarrow \forall i \leq n, x_i \vee y_i \neq \top_i$. A subset X of D is properly compatible (notation $\Uparrow X$) if $\forall$ $x,y \in X$, $x \Uparrow y$. ∎

2.6 definition:given a basic SK-domain D, an element x of D is said to be *prime* if :

$\forall X \subseteq D, ((\Uparrow X) \wedge (x \leq \bigvee X)) \Rightarrow (\exists \alpha \in X: x \leq \alpha)$. ∎

Notice that the basic SK-domains are not only ω-algebraic lattices but also ω-prime-algebraic lattices ie: any element dominates a denumerable set of prime elements and is the lub of the prime elements it dominates. That is to say, basic SK-domains have a particular base. Henceforth, given a basic domain D, we shall refer to the set of its prime as $\mathbf{Pr}(D)$, while $\mathbf{Pr}(x)$

will represent the set of prime elements dominated by x. Prime algebraic domains are well known structures; see for example [1] where they allow a nice characterization of stable functions. Their properties constitute an important part of the theories concerning the concrete domains defined by Khan and Plotkin: see for example [6] where prime algebraic lattices are related to event structures and also characterized in terms of distributivity.

The basic intensionnal equivalence relation is defined as follows:

2.7 __definition__: let a and b be two finite elements of a basic domain D; then a is *intensionally equivalent* to b (notation: $a \sim b$) if: $\forall c \in \mathbf{Pr}(D)$, $(a \neq c \ \& \ a \Uparrow c) \Leftrightarrow (b \neq c \ \& \ b \Uparrow c)$ ■

Intuitively $a \sim b$ can be interpreted as follows: considering any piece of information (here represented by c), if it can be used to increase significantly the information contained in a then it can be made so with b and vice versa. Since individuals can be considered as sets of pieces of information, it is natural to set two elements equivalent as soon as they contain the same equivalence classes of finite elements. Further, if a prime element is viewed as filling a place in an element then two elements should be equivalent if they provide the same places to be filled. This idea, when transposed to the basic level, leads to the above definition.

Let us verify that this relation is actually an equivalence relation:

2.8 __proposition__: $\sim$ is an equivalence relation over $\mathfrak{B}(D)$.
__proof__: reflexivity and symetry are immediate. Let us show transitivity; let $a, b, c \in \mathfrak{B}(D)$ satisfying: $a \sim b \sim c$; we have to show $a \sim c$. But, $\forall d \in \mathbf{Pr}(D)$, $(a \neq d \ \& \ a \Uparrow d) \Leftrightarrow (b \neq d \ \& \ b \Uparrow d) \Leftrightarrow (d \neq c \ \& \ d \Uparrow c)$; qed. ■

We now illustrate on an example the relation between the notion of place and our definition. We want to show here, that our intensional preorder allows to verify whether or no two tuples are defined on the same components.

2.9 __proposition__: Define domain $\mathbf{D}=(D)^k$ where $k \in \mathbb{N}$ and D is any flat lattice based SK-domain, with an intensional relation defined as above. Let the elements of $\mathbf{D}$ be ordered componentwise. Then two elements $x=\langle x_i \rangle_{i \leq k}$, $y=\langle y_i \rangle_{i \leq k}$ in $\mathbf{D}$ are defined on the same components iff $\mathbf{Pr}(x)/\sim = \mathbf{Pr}(y)/\sim$.
__proof__ It is immediate that $\mathbf{D}$ is a basic SK-domain with as prime elements, those sequences which contain only undefined elements except possibly for a unique component (which has to be prime). Prime elements that differ from the undefined element can thus be represented by pairs (p,v), where $1 \leq p \leq k$ et $v \in D$.

Now we claim: if $a=(p_a,v_a)$ and $b=(p_b,v_b)$ are two prime and equivalent elements, both different from $\perp$, then $p_a=p_b$.
Indeed, let $c=(p,v)$ be a prime element; then it is easily seen that $(a \Uparrow c \ \wedge \ a \neq c) \Leftrightarrow ((p_a=p) \Rightarrow$

$(v_a=v)) \wedge ((p_a \neq p \vee v_a \neq v))$; ie: $a \sim b \Rightarrow \forall p, p \neq p_a \Leftrightarrow p \neq p_b$ and thus $p_a = p_b$ .The result follows ∎

The above proposition means that one can talk about arguments' position in a list of arguments without any references to the product structure of domains.

Before we proceed, let us introduce some examples:

<u>**example1**</u>: Consider the domain $\mathbb{N}$ ordered by $n \leq m$ iff $m=\top$ <u>or</u> $n=m$ <u>or</u> $n=\perp$ then all numbers are equivalent: $\forall x,y \quad (\top \neq x \neq \perp \neq y \neq \top) \Rightarrow x \sim y$.

<u>**example2**</u>: Let $D = \mathbf{T}^2$ orderd with the product order, where $\mathbf{T}$ is the flat lattice $\{\perp, tt, ff, \top\}$; then: i) $(\perp,\perp) \lesssim (tt, \perp) \sim (ff, \perp)$.

ii) $(tt, \perp) \not\lesssim (\perp, tt)$, since $(tt, \perp) \neq (\perp, ff) \neq (\perp, tt)$ & $(\perp, ff) \Uparrow (tt, \perp)$ but $\neg \ (\perp, ff) \Uparrow (\perp, tt)$

Notice that our definition illuminates the fact that at the basic level, the extensional partial order of SK-domains is actually made of to separate relations: on one hand the intensional preorder relation and on the other hand the proper compatibility relation:

2.10 <u>**lemma**</u>: Let D be a basic SK-domain then:

i) $\forall a,b \in \mathcal{P}r(D)$, $(a \Uparrow b \wedge a \sim b) \Leftrightarrow a=b$

ii) $\forall x,y \in D$, $(x \lesssim y \ \& \ x \Uparrow y) \Leftrightarrow x \leq y$.

<u>proof</u>: i) suppose $a \neq b$ then since $a \Uparrow b$, we get the contradiction $b \neq b$.

point ii) is an easy corrolary of point i) ∎

We now turn to the general case and we assume that all SK-domains are constructed from basic ones via the type hierarchy of typed λ-calculus (ie using product and exponentiation).
We consider function spaces explicitely below. In the case of product spaces, we define the intensional relations to be the product relation.

2.11 <u>**definition**</u>: let $< D, \lesssim_1, \sim_1 >$ and $< E, \lesssim_2, \sim_2 >$ be two SK-domains and let φ and ψ be two finite elements of the domain ($[D \rightarrow E]$) of continuous functions from D to E; then φ and ψ are *intensionally equivalent* (notation $\varphi \sim \psi$) if $\varphi \neq \perp \neq \psi$, where $\perp$ denotes the totally undefined function. ∎

Notice that this is a significant variation compared with our previous versions. Indeed this definition amounts to treat functions as first class citizens, and thus to consider that they occupy a unique place in an argument list. Actually we consider that the parameters of a function(al) are structured as a list of pointers, each pointer pointing to the value of one parameter (null pointer represents absence of parameter). With that hypothesis, one only has to check that two lists of parameters have corresponding non null pointers (regardless to the value of the parameters), in order to declare them intensionally equivalent.

Let us also remark that the functional intensional relation is strictly coarser than the pointwise extension of the basic one: if φ and ψ are two finite elements in $[D \rightarrow E]$ s.t $\forall x \in D$, $\varphi(x) \sim \psi(x)$, then $\varphi \sim \psi$; the converse is false in general.

Henceforth if $X, Y \subseteq D$, we write "$X \lesssim Y$" to mean: $\forall\ \alpha \in X,\ \exists \beta \in Y:\ \alpha \lesssim \beta$ (same thing with extensional preorder -or its negation- substituted for the intensional one -or its negation-). Same convention holds with the strict relations; for example: "$X<Y$" means: $X \leq Y$ & $X \neq Y$. Further, we write x, for singleton $\{x\}$, except where ambiguous; for example if $x \in D$, then "$x \lesssim X$" means: $\exists\ \alpha \in X:\ x \lesssim \alpha$.

III) <u>S-FUNCTIONS</u>

First of all, let us recall that a function f from domain D to domain E is continuous if: $\forall x \in D, \forall \beta \in E\ (\ \beta \text{ isolated } \& \ \beta \leq f(x)\ ) \Rightarrow (\ \exists\ \alpha \in D : \alpha \text{ isolated } \& \ \alpha \leq x \ \& \ \ \beta \leq f(\alpha)\ )$.
Equivalently f is continuous iff it preserves lubs of directed subsets.

Considering our domains, the subset of finite elements dominated by an element x is directed and has lub x. That is to say, a continuous function is completely characterized by its values on the finite elements. Therefore, while considering continuous functions, we shall restrict to finite elements only, without any lost of generality.

We now introduce S-functions.
Given two SK-domains D and E, we are interested in those continuous functions from D to E satisfying the following condition (S) :

$$\forall\ x \in D, \forall\ \beta \in E,\ (\ \beta \not\leq f(x)\ \&\ \exists\ z \in D:\ z>x\ \&\ \beta \leq f(z)\ ) \Rightarrow \exists\ \omega(f,x,\beta) \in D:$$

(S): $\quad$ i) $\omega(f,x,\beta) \not\leq x \ \ \& \ \ \omega(f,x,\beta) \leq z$

$\quad$ ii) $\forall\ y \in D,\ (\ y>x\ \ \&\ \ \beta \lesssim f(y)\ ) \Rightarrow \omega(f,x,\beta) \lesssim y$

Intuitively, a function verifies condition (S) if any increase of information in its result necessitates the increase of an intensionally predefined information in its argument. Indeed this idea is a generalization of Kahn and Plotkin's definition of sequential functions, in particular when considering basic domains, those where relation $\beta \lesssim x$ can be interpreted as:" β represents places in x". By the way, $\omega(f,x,\beta)$ can be viewed as *an index of f in x for value* β; we then remark as already noticed by Vuillemin, that an index is not necessarily unique. It is worth noticing that our index not only depends on parameters (f,x,β), but also is given a specialized direction namely, z here. Thus a more adequate name would be $\omega(f,(x,z),\beta)$ instead (but, let us preserve "traditions").

Henceforth, we write $[D -_S\!\!\rightarrow E]$ for the set of continuous functions from D to E which verify (S) and we call them S-functions.

Before we examine S-functions further, let us verify that SK-domains together with S-functions form a category.

3.1 proposition: the identity function $1_D \in [D \rightarrow D]$: $\forall x \in D,\ 1_D(x)=x$, is an S-function.

proof: one immediately verifies that we can put $\omega(1_D,x,\beta)=\beta$ ■

3.2 proposition: $\forall f \in [D -_S\!\!\rightarrow C], \forall g \in [C -_S\!\!\rightarrow E], g \circ f \in [D -_S\!\!\rightarrow E]$.

proof: let $x \in D$, $\beta \in E$ s.t: $\beta \not\leq g\circ f(x)\ \&\ \exists z \in D: z>x\ \&\ \beta \leq g\circ f(z)$ then $\exists\, \omega(g,f(x),\beta) \in C$:

i) $\omega(g,f(x),\beta) \not\leq f(x)\ \&\ \omega(g,f(x),\beta) \leq f(z)$

ii) $\forall y \in C,\ (y>f(x)\ \&\ \beta \leq g(y)) \Rightarrow \omega(g,f(x),\beta) \leq y$

and thus $\exists\, \omega(f,x,\omega(g,f(x),\beta)) \in D$:

i) $\omega(f,x,\omega(g,f(x),\beta)) \not\leq x\ \&\ \omega(f,x,\omega(g,f(x),\beta)) \leq z$

ii) $\forall y \in D,\ (y>x\ \&\ \beta \leq g\circ f(y)) \Rightarrow \omega(g,f(x),\beta) \leq f(y) \Rightarrow \omega(f,x,\omega(g,f(x),\beta)) \leq y$. ■

We can thus speak of the category whose objects are SK-domains and whose arrows are S-functions. We call it SKD. Our intent is to show that SKD is a Λ-category.

Before, let us examine more closely S-functions.

Fact 1: *S-functions different from identity do exist*

proof: an example of S-function is the following $f: \mathbf{T} \rightarrow \mathbf{T}$ defined by: $f(\bot)=\bot$, $f(tt)=tt=f(ff)=f(\top)$. Indeed, it is easily verified that $\omega(f,\bot,tt)$ can be set to tt.

Another example of the existence of S-functions is given by the following

3.3 proposition: let d and e be two isolated element of an SK-domaine D; then the function f denoted $(d \Rightarrow e)$ s.t: $f(x)=$ if $x \geq d$ then e else $\bot$, is an S-function from D to E.

proof: function f is continuous since d and e are isolated. Moreover, $\omega(f,x,\beta)$ is definable only if $\beta \leq e$ and $\neg(x \geq d)$ and $(\exists z>x : z \geq d)$; but in that case one immediately verifies that $\omega((d \Rightarrow e),x,\beta)$ can be set to d. ■

An easy corollary of the above is that constant functions are S-functions. This result can be proved directly however, since the premises of the existence of indexes cannot be verified with constant functions.

Fact 2: *S-functions are not divided in strict versus constant continuous functions*
proof: the function f: $\mathbf{T}^2 \to \mathbf{T}^2$ defined by: f(x)= if $x \geq (\perp,tt)$ then (tt,tt) else (tt,⊥), is an S-function which is neither strict nor constant.

Fact 3: *continuous functions strictly contain S-functions*:
proof: we exhibit two continuous functions which are not S-functions:

i) the paradigm of non sequential function the "parallel-or", is not an S-function since $\omega(por,(\perp,\perp),tt)$ does not exist; indeed $(\perp,tt) \not\lesssim (tt,\perp) \not\lesssim (\perp,tt)$.

ii) another example of non S-function is due to Berry: define function perm:$\mathbf{T}^3 \to \mathbf{T}$ to be the least continuous function st: perm(tt,ff, ⊥)=perm(ff, ⊥,tt)=perm(⊥,tt,ff)=tt. Then remark that $\omega(perm, (\perp,\perp,\perp),tt)$ does not exist.

Fact 4: *absence of indexes is not nessarily detected in ⊥*
proof: define f to be the least continuous function from $\mathbf{T}^3$ to $\mathbf{T}$ s.t: f(x)=tt if $x \geq (tt,ff,\perp)$ or $x \geq (tt,\perp,ff)$; then f is not an S-function since $\omega(f, (tt, \perp,\perp),tt)$ does not exist thought $\omega(f,(\perp,\perp,\perp),tt)$ does exist.

T.Ehrhard and A. Bucciarelli, pointed to us that this phenomenum enhance the fact that indexes are necessary but not sufficient informations for increasing the result of a function: For example, let us consider the following example (due to T.Ehrhard and A. Bucciarelli) : let f be the least continuous function from $\mathbf{T}^3$ to $\mathbf{T}$ such that: f(x)=tt iff $x \geq (tt,tt,\perp)$ or $x \geq (ff,\perp,tt)$; then f is intuitively sequential (and is indeed an S-function) but it does not exist any element ε such that $tt \lesssim f(\varepsilon)$ and which could be put for $\omega(f, (\perp,\perp,\perp),tt)$) .

To synthesize, the preceding facts and remarks make the point that in general, it is not true that S-functions verify $\beta \leq f(\omega(f,x,\beta))$. However, restricting to S-functions which have that property leads to a subclass which can be nicely characterized:

3.4 definition: given SK-domains D and E, a continuous function f from D to E is said to be *strongly sequential* if it verifies the following condition (S_f) :

$$\forall x \in D, \forall \beta \in E, \ (\beta \not\leq f(x) \ \& \ \exists z \in D \text{ tq: } z > x \ \& \ \beta \leq f(z)) \Rightarrow \exists \Omega(f,x,\beta) \in D:$$

$$(S_f): \quad \text{i) } \beta \leq f(\Omega(f,x,\beta)) \ \& \ \Omega(f,x,\beta) \leq z$$

$$\text{ii) } \forall y \in D, \ (y > x \ \& \ \beta \lesssim f(y)) \Rightarrow \Omega(f,x,\beta)) \lesssim y$$

It is immediate that S_f-functions constitute a subclass of S-functions; moreover, they are characterized by the fact that all significant indexes can be exhibited in ⊥, as indiquated in the following:

3.5 proposition: a continuous function $f \in [D \to E]$ verifies (S_f) iff it verifies the following condition

$(S_\perp)$: $\forall \beta \in E$, $(\exists z \in D: \beta \leq f(z)) \Rightarrow \exists\, \omega(f,\beta) \leq z: \beta \leq f(\omega(f,\beta))\ \&\ \forall y \in D, \beta \lesssim f(y) \Rightarrow \omega(f,\beta) \lesssim y$.

proof: $(S_f) \Rightarrow (S_\perp)$: let $f \in [D\text{-}_{S_f} \to E]$ and $\beta \in E$ s.t: $\beta \leq f(D)$; two cases are considered:

i) $\beta \leq f(\perp)$; this case is impossible since $\exists\, x: \beta \not\leq f(x)$.

ii) $\beta \not\leq f(\perp)$; in this case one immediately verifies that $\omega(f,\beta)$ can be identified with $\Omega(f,\perp,\beta)$.

$(S_\perp) \Rightarrow (S_f)$: let f verify $(S_\perp)$ and suppose x and β such that the premises of (S_f) hold; we claim that $\Omega(f,x,\beta)$ exists and can be set to $\omega(f,\beta)$; indeed $\omega(f,\beta)$ exists by hypothesis; moreover, by definition, $\beta \leq f(\omega(f,\beta))\ \&\ \forall y \in D, \beta \lesssim f(y) \Rightarrow \omega(f,\beta) \lesssim y$. qed ■

IV) THE Λ- CATEGORY SKD

We first examine the S-functions space between SK-domains. Our intent is to show that this space is an SK-domaine.

For the extensional partial order, we use Scott's order : $f \leq g \Leftrightarrow \forall x, f(x) \leq g(x)$.

The intensional preorder is generated by the intensional equivalence we already defined on functional finite elements.

A first result of this section is the following:

4.1 theorem: the function space of S-functions between SK-domains is an SK-domain where glb of arbitrary subsets are taken pointwise.

The proof of this result is an easy corollary of proposition 3.3 and the followings :

4.2 proposition: let $(f_i)_i \subseteq [D\text{-}_S \to E]$, then $\wedge_i (f_i) \in [D\text{-}_S \to E]$.

proof: first, notice that the constant functions $\lambda x.\perp$ and $\lambda x.\top$ which are the glb of the whole domain and of the empty subset, respectively, are S-functions.

Now let $(f_i)_i$ be a non empty subset of S-funtions and let $x \in D$ and $\beta \in E$ be s.t:

$\beta \not\leq \wedge_i (f_i)(x)\ \&\ \exists\, z \in D: z > x\ \&\ \beta \leq \wedge_i (f_i)(z)$ then

$\forall i, \beta \leq f_i(z)$ and $\exists\, j: \beta \not\leq f_j(x)$ and thus $\exists\, \omega(f_j,x,\beta) \in D$:

i) $\omega(f_j,x,\beta) \not\leq x\ \&\ \omega(f_j,x,\beta) \leq z$

ii) $\forall y \in D$, $(y > x\ \&\ \beta \lesssim \wedge_i (f_i)(y)) \Rightarrow \beta \lesssim f_j(y) \Rightarrow \omega(f_j,x,\beta) \lesssim y$

ie: $\omega(\wedge_i(f_i),x,\beta)$ can be set to $\omega(f_j,x,\beta)$. qed ▪

We now turn to the categorical properties of SKD.

First of all, recall that a Λ-category is a cartesian closed category which is order-enriched (ie: the homsets are provided with order relations such that the composition operation is continuous), together with some additional continuity properties.

In [1] Berry has shown that in order to still have a Λ-category while restricting Scott's Λ-category by imposing condition P to objects and condition Q to arrows (this is our case indeed), one has to verify the following conditions:

1) composition closedness: if D,E,F verify P and if $f:D\to E$ & $g:E\to F$ verify Q, then $g\circ f:D\to F$ verify Q; moreover, the identity arrows of objects that verify P, verify Q.

2) product closedness: if D and E verify P, then $D\times E$ verify P. The projection functions from $D\times E$ to D and to E verify Q. If F verify P, and if $f: F\to D$ & $g:F\to E$ verify Q, then the function $<f,g>: F\to D\times E$ verifies Q (idem for the denumerable case).

3) exponentiation closedness: if D and E verify P, then the set $[D\!-_Q\!\to E]$ (of functions from D to E that verify Q) can be provided with an order ≤ such that $< [D\!-_Q\!\to E], \leq >$ verifies P, and such that the function"app" which applies functions to their arguments verifies Q and finally such that the curryfied form of a function between objets that verify P, verifies Q, as soon as the function verifies Q.

4) continuity properties: the functions of composition, of pair formation and of curryfication are continuous.

First we show the following :

4.3 theorem: SKD is cartesian closed.

proof: let us notice that we have a little bit less to do here since we use the extensional order and thus, we need not worry about continuity. Indeed we just have to verify product and exponentiation closednesses in order to ensure that SKD is cartesian closed.

1) product closedness:

i) the cartesian product of a finite (or denumerably infinite) family of SK-domains is an SK-domain.

This is the case for domains, with as finite elements of the product domain those sequences of undefinite elements except possibly one finite element, and with as partial order, the product of the orders of the domains. Now to get an SK-domain, we define the intensional preorder to be the product of the preorders of the domains.

ii) the projection functions π_i from $D_1\times D_2$ to D_i for $i\in\{1,2\}$, are S-functions:

It is easy to verify that for any $\beta_i \in D_i$, $i \in \{1,2\}$ and any $<x,y> \in D_1 \times D_2$,

$\omega(\pi_1,<x,y>,\beta_1) = <\beta_1, \perp>$ and $\omega(\pi_2,<x,y>,\beta_2) = <\perp, \beta_2>$,

iii) if $f \in [F -_S \rightarrow D]$ and $g \in [F -_S \rightarrow E]$ then $<f,g> \in [F -_S \rightarrow D \times E]$.

verify that $\omega(<f,g>,x,\beta)$ can be set to:

i) $\omega(f,x,\pi_1(\beta))$ if $\pi_1(\beta) \not\leq f(x)$ & $\pi_2(\beta) \leq g(x)$

ii) $\omega(g,x,\pi_2(\beta))$ if if $\pi_2(\beta) \not\leq g(x)$ & $\pi_1(\beta) \leq f(x)$

iii) to either $\omega(f,x,\pi_1(\beta))$ or $\omega(g,x,\pi_2(\beta))$, if both $\pi_1(\beta) \not\leq f(x)$ and $\pi_2(\beta) \not\leq g(x)$.

2) <u>exponentiation closedness:</u>

i) function app:$<[D -_S \rightarrow E] \times D> \rightarrow E$ is an S-fonction

let $<f,x> \in <[D -_S \rightarrow E] \times D>$, $\beta \in E$ and let $<h,z> \in <[D -_S \rightarrow E] \times D>$ s.t:

$(\beta \not\leq f(x))$ & $(<f,x> < <h,z>)$ & $(\beta \leq h(z))$; let us exhibit $\omega(app,<f,x>,\beta)$ s.t:

- $\omega(app,<f,x>,\beta) \in <[D -_S \rightarrow E] \times D>$ &
- $\omega(app,<f,x>,\beta) \not\leq <f,x>$ & $\omega(app,<f,x>,\beta) \leq <h,z>$
- $\forall$ $<g,y> \in <[D -_S \rightarrow E] \times D>$, $(<f,x> < <g,y>$ & $\beta \leq g(y)) \Rightarrow \omega(app,<f,x>,\beta) \leq <g,y>$.

It is then enought to set $\omega(app,<f,x>,\beta) = <h,x>$; and this is possible since all non undefined functions are intensionally equivalent.

ii) function curry :$[D \times E -_S \rightarrow F] \rightarrow [D \rightarrow [E \rightarrow F]$ is an S-isomorphisme of SK-domains:

consider an S-function f from $D \times E$ to F:

a) we show that the function f_a , from E to F, s.t:$\forall$ $b \in E$, $f_a(b) = f(a,b)$, verifies (S).

Let $x, z \in E$ & $\beta \in F$ and suppose $z > x$, $\beta \not\leq f_a(x)$ & $\beta \leq f_a(z)$, then we have:

$(a,x), (a,z) \in D$ & $\beta \in F$ & $(a,z) > (a,x)$ & $\beta \not\leq f(a,x)$ & $\beta \leq f(a,z)$; thus $\exists$ $\omega(f,(a,x),\beta)$ s.t:

$\omega(f,(a,x),\beta) \not\leq (a,x)$ & $\omega(f,(a,x),\beta) \leq (a,z)$ & $\forall$ $(a,y) > (a,x)$, $\beta \leq f(a,y) \Rightarrow \omega(f,(a,x),\beta) \leq (a,y)$.

That is to say, we can set $\omega(f_a,x,\beta) = \pi_2(\omega(f,(a,x),\beta))$.

b) we show that the function $c(f)$ from D to $[E -_S \rightarrow F]$ and s.t:$\forall$ $x \in D$, $c(f)(x) = f_x$, is an S-function.

Let $x, z \in D$ & $\beta \in [E -_S \rightarrow F]$ be s.t: $z > x$ & $\beta \not\leq c(f)(x)$ & $\beta \leq c(f)(z)$, then:

$\exists$ $(x,e) \in D \times E$: $\beta(e) \not\leq f(x,e)$, & $\exists$ $(z,e) \in D \times E$ st: $\beta(e) \leq f(z,e)$. That is to say $\omega(f,(x,e),\beta(e))$ exists;

we then just set: $\omega(c(f),x,\beta) = \pi_1(\omega(f,(x,e),\beta(e))))$.

c) let us examine function curry .

we have to prove: $\forall$ $f,h \in [D \times E -_S \rightarrow F]$, $\forall$ $\beta \in [D -_S \rightarrow [E -_S \rightarrow F]]$, if $f < h$ & $\beta \not\leq c(f)$ & $\beta \leq c(h)$, then $\exists$ $\omega(curry,f,\beta) \in [D \times E -_S \rightarrow F]$ s.t:

$\omega(curry,f,\beta) \not\leq f$ & $\omega(curry,f,\beta) \leq h$ & $\forall$ $g > f$, $\beta \leq c(g) \Rightarrow \omega(curry,f,\beta) \leq g$.

Indeed, one verifies that $\omega(\text{curry},f,\beta) = \vee\{((\delta,\varepsilon) \Rightarrow \phi) / (\delta \Rightarrow (\varepsilon \Rightarrow \phi) \leq \beta\}$.
As a corollary, we conclude that curry is an S-isomophism of SK-domains. ■

We now introduce our most significant result:

4.4 theorem: SKD is an order extensional Λ-category.
proof: corollary of results 4.1 & 4.3. ■

V) CONCLUSION

The goal of this paper was to close the sequentiality question. Our opinion is that we succeed, at least in the first part of the problem: we provide here, a concept of sequential continuous function which can be used to define an order extensional model of typed λ-calculus (equivalently, a cartesian closed category enriched with Scott's order).

As for the full abstraction problem (which indeed raised the sequentiality question), we think that our approach of relaxing the constraints imposed by the intensionnal preorders on domains, will probably allows to get more insight in the real natureof the problem. Let us comment the situation a little bit: the trick that makes things behave well here, is that we have chosen the intuitively coarsed intensional relation one would like to impose, without reaching triviality, and without supplementary coherence constraints to manage. Of course the resulting model is not Milner's semantical counterpart since the "Currien's counterexamples" still hold! However, the usage of "free-taste" intensionnal preorders could allow to progressively restricts the preorders so as to eliminate conterexamples while keeping cartesian closedness. Following that way we are not be obliged to work with quotient models, and this will give more insight in the structure of the elment of the model. That is the matter of ongoing works.

Let us enhance the fact that the domains we used here are not the traditional cpos, but rather lattices.This restitution of the over defined element is not really necessary, since similar results can be get with cpo-based domains (see[4] for example).

Another point to discuss here, is the relation between S-functions and Berry's stable functions. It is already known that stability is different from sequentiality (recall the "perm" function of section III). Moreover, we notice here that sequentiality does not even induce stability (recall the "def" function in the introduction). Of course it would be very nice to show that S-functions are stable; however it is clear as well that this result cannot hold with our extensional order on domains since Berry has already shown that the exponentiation operation is not stable when Scott's order is used. We think that this failure is essentially due to the non distributivity of the domain of sequential functions: for example, considering the domain $[\mathbf{T}^2 \text{-}_S\!\!\rightarrow \mathbf{T}]$, let
$f_1 = (\perp,tt) \Rightarrow tt$, $f_2 = (tt,\perp) \Rightarrow tt$ and $g = (ff,ff) \Rightarrow tt$; then $f_1 \vee f_2 = \lambda x.tt$ and

$g \wedge (f_1 \vee f_2) = g$ but $(g \wedge f_1) \vee (g \wedge f_2) = \lambda x.\perp$.

We conjecture that some kind of intensional stability can subsist however; the question is to exhibit this form of stability since in general it cannot be reduced to: $\forall f \in [D \text{-}_s\!\!\to E]$, $\forall x \in D, \forall \beta \in E$, $\beta \leq f(x) \Rightarrow \exists\ m(f,x,\beta) \in D$ s.t: $(m(f,x,\beta) \leq x)$ & $(\forall y \leq x,\ \beta \leq f(y) \Leftrightarrow m(f,x,\beta) \leq y)$.

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